

# THE FREE ENERGY IN A CLASS OF QUANTUM SPIN SYSTEMS AND INTERCHANGE PROCESSES

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**ABSTRACT.** We study a class of quantum spin systems in the mean-field setting of the complete graph. For spin  $S = \frac{1}{2}$  the model is the Heisenberg ferromagnet, for general spin  $S \in \frac{1}{2}\mathbb{N}$  it has a probabilistic representation as a cycle-weighted interchange process. We determine the free energy and the critical temperature (recovering results by Tóth and by Penrose when  $S = \frac{1}{2}$ ). The critical temperature is shown to coincide (as a function of  $S$ ) with that of the  $q = 2S + 1$  state classical Potts model, and the phase transition is discontinuous when  $S \geq 1$ .

## 1. INTRODUCTION

It has been well-known since the work of Tóth [23] and Aizenman and Nachtergaele [1] in the early 1990's that many quantum spin-systems can be analyzed using probabilistic representations. Tóth's representation of the (spin- $\frac{1}{2}$ ) Heisenberg ferromagnet in terms of random transpositions is particularly appealing in its simplicity. However, though simple to define, it has proved challenging to obtain rigorous results using this representation. While substantial progress has been made on several other models using probabilistic representations [6, 7, 9, 10, 13, 14, 18, 24], proving a phase-transition in the ferromagnetic Heisenberg model on the lattice  $\mathbb{Z}^d$  remains an open challenge.

For mean-field variants there has been more progress, and related models have recently received quite a lot of attention in the probability literature [2, 3, 4, 5, 8, 16, 17, 21]. The free energy of the spin- $\frac{1}{2}$  Heisenberg ferromagnet on the complete graph was determined already in 1990: by Tóth [22] using a random-walk representation, and simultaneously but independently by Penrose [19] by explicitly diagonalizing the Hamiltonian.

Here we extend the latter results to a class of spin  $S \in \frac{1}{2}\mathbb{N}$  models, with Hamiltonian equal to a sum of transposition-operators (see below for a precise definition). Probabilistically, the model naturally generalizes Tóth's permutation-representation: a weight factor  $2^{\#\text{cycles}}$  is replaced by  $(2S + 1)^{\#\text{cycles}}$ . Our approach is different both from that of Tóth and that of Penrose. The key step is to obtain an expression for

the partition function in terms of the irreducible representations of the symmetric group. Perhaps our most surprising result is a connection to the classical Potts-model: we show that the critical temperature of our model, as a function of  $S$ , coincides with that of the  $q = 2S + 1$  state Potts model.

We now define the model and state our primary results.

**1.1. Model and main results.** We let  $S^1, S^2, S^3$  denote the usual spin-operators, satisfying the relations

$$[S^1, S^2] = iS^3, \quad [S^2, S^3] = iS^1, \quad [S^3, S^1] = iS^2,$$

where  $i = \sqrt{-1}$ . For each  $S \in \frac{1}{2}\mathbb{N}$  we work with the standard spin- $S$  representation, where the  $S^j$  are Hermitian matrices acting on  $\mathcal{H} = \mathbb{C}^{2S+1}$ . We fix an orthonormal basis for  $\mathcal{H}$  consisting of eigenvectors for  $S^3$ , denoting the basis vector with eigenvalue  $a \in \{-S, -S+1, \dots, S\}$  by  $|a\rangle$ .

Let  $G = K_n = (V, E)$  be the complete graph on  $n$  vertices, i.e. the graph with vertex set  $V = \{1, \dots, n\}$  and edge set  $E = \binom{V}{2}$  consisting of one edge (bond) per pair  $x \neq y$  of vertices. For each  $x \in V$  we take a copy  $\mathcal{H}_x$  of  $\mathcal{H}$ , and we form the tensor product  $\mathcal{H}_V = \otimes_{x \in V} \mathcal{H}_x$ . An orthonormal basis for  $\mathcal{H}_V$  is given by the vectors  $|\mathbf{a}\rangle = \otimes_{x \in V} |a_x\rangle$  for  $\mathbf{a} = (a_x)_{x \in V} \in \{-S, \dots, S\}^V$ . If  $A$  is an operator acting on  $\mathcal{H}$  we define  $A_x$  acting on  $\mathcal{H}_V$  by  $A_x = A \otimes \text{Id}_{V \setminus \{x\}}$ .

The transposition operator  $T_{xy}$  on  $\mathcal{H}_V$  is defined as follows. For each pair  $x \neq y$  of vertices,  $T_{xy}$  is given by its action on the basis elements  $|\mathbf{a}\rangle$ :

$$(1) \quad T_{xy} \otimes_{z \in V} |a_z\rangle = \otimes_{z \in V} |a_{\tau(z)}\rangle,$$

where  $\tau = (x, y)$  is the transposition of  $x$  and  $y$ :

$$\tau(z) = \begin{cases} y, & \text{if } z = x, \\ x, & \text{if } z = y, \\ z, & \text{otherwise.} \end{cases}$$

Thus  $T_{xy}$  interchanges the  $x$  and  $y$  entries of  $|\mathbf{a}\rangle$ .

Our model has the Hamiltonian

$$(2) \quad H = H_n = - \sum_{xy \in E} (T_{xy} - 1)$$

acting on  $\mathcal{H}_V$ . We take the inverse-temperature of the form  $\beta/n$  for constant  $\beta > 0$ , thus the partition function is

$$(3) \quad Z_n(\beta) = \text{tr}(e^{-(\beta/n)H_n}).$$

We note that  $T_{xy}$  may be expressed as a polynomial in the operators  $\mathbf{S}_x \cdot \mathbf{S}_y = \sum_{j=1}^3 S_x^j S_y^j$ . For example, when  $S = \frac{1}{2}$  we have that  $T_{xy} = 2(\mathbf{S}_x \cdot \mathbf{S}_y) + \frac{1}{2}$ , and when  $S = 1$  that  $T_{xy} = (\mathbf{S}_x \cdot \mathbf{S}_y)^2 + (\mathbf{S}_x \cdot \mathbf{S}_y) - 1$ . (See Proposition A.1 in the appendix for the general case.) Thus for

$S = \frac{1}{2}$  we recover the Heisenberg ferromagnet at inverse-temperature  $2\beta/n$ .

Our first main result is an explicit formula for the free energy. For each  $S \in \frac{1}{2}\mathbb{N}$ , let  $\theta = 2S + 1$  and let

$$\Delta = \Delta_\theta = \{x = (x_1, \dots, x_\theta) \in [0, 1]^\theta : x_1 \geq \dots \geq x_\theta, \sum_{j=1}^\theta x_j = 1\}.$$

Define the function  $\phi_\beta : \Delta \rightarrow \mathbb{R}$  by

$$(4) \quad \phi_\beta(x) = \frac{\beta}{2} \left( \sum_{j=1}^\theta x_j^2 - 1 \right) - \sum_{j=1}^\theta x_j \log x_j.$$

THEOREM 1.1. *We have that*

$$(5) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\beta) = \max_{x \in \Delta} \phi_\beta(x).$$

As mentioned previously, our analysis relies on a probabilistic representation. We describe this now. Let  $\mathbb{P}(\cdot)$  be a probability measure governing a collection  $\omega = (\omega_{xy} : xy \in E)$  of independent rate 1 Poisson processes on  $[0, \beta/n]$ , indexed by the edges of  $G$ . Thus each  $\omega_{xy}$  is a random (almost-surely finite) subset of  $[0, \beta/n]$ ; the number of elements of  $\omega_{xy}$  in an interval  $[s, t]$  has the Poisson distribution  $\text{Po}(t - s)$ , and these numbers are independent for disjoint intervals. We think of  $[0, \beta/n]$  as a time-interval. See Figure 1 for a pictorial representation.

As explained in e.g. [1, eq. (2.11)] we have from the Lie–Trotter product formula that

$$(6) \quad e^{-(\beta/n)H_n} = \mathbb{E} \left[ \prod_{(xy,t) \in \omega}^* T_{xy} \right],$$

where  $\Pi^*$  is the time-ordered product over all elements of  $\omega$ . In light of (1) and (6) we may think of each point  $(xy, t) \in \omega$  as representing a transposition of  $x, y \in \{1, \dots, n\}$  at time  $t$ . We let  $\sigma = \sigma(\omega) = \prod_{(xy,t) \in \omega}^* (x, y)$  denote the (time-ordered) composition of these transpositions from time 0 to time  $\beta/n$ . Thus  $\sigma \in \mathcal{S}_n$ , the symmetric group on  $n$  letters.

Recall that each  $\sigma \in \mathcal{S}_n$  may be written as a product of disjoint cycles (orbits). Let  $\ell = \ell(\omega)$  denote the number of such cycles of  $\sigma(\omega)$ , including singletons. Taking the trace in (6) we find that we get a contribution of 1 from each basis vector  $|\mathbf{a}\rangle$  for which the function  $\mathbf{a} : V \rightarrow \{-S, \dots, S\}$  is constant on each cycle of  $\sigma(\omega)$ . (Figure 1 is helpful in verifying this statement.) From the other  $|\mathbf{a}\rangle$  we get contribution 0. Writing  $\theta = 2S + 1$ , as before, for the number of possibilities per cycle, we conclude that

$$(7) \quad Z_n(\beta) = \text{tr}(e^{-(\beta/n)H_n}) = \mathbb{E}[\theta^{\ell(\omega)}].$$

In order to identify a phase-transition we will work also with a ‘weighted’ version of (7). Let  $\mathcal{C} = \mathcal{C}(\omega)$  denote the set of cycles of

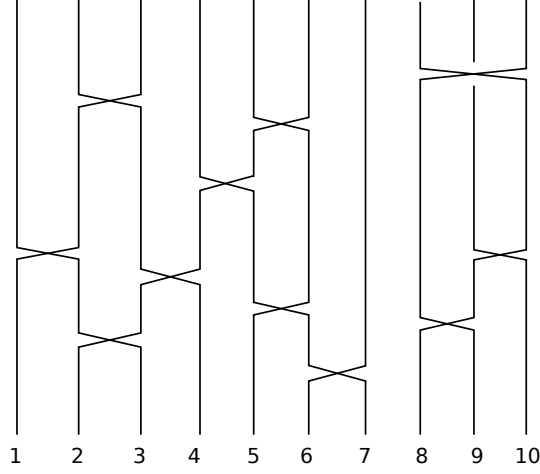


FIGURE 1. A sample  $\omega$ , with the vertex set  $V = \{1, \dots, 10\}$  on the horizontal axis and time going upwards. Elements  $(xy, t) \in \omega$  are represented as crosses, and are to be thought of as transpositions. (In this picture, for clarity only, most crosses occur between consecutive vertices.) Here  $\sigma(\omega) = (1, 3)(2, 6, 7, 4)(9, 10)(5)(8)$  and  $\ell(\omega) = 5$ .

the permutation  $\sigma(\omega)$ , and for  $h \in \mathbb{R}$  write

$$(8) \quad Z_n(\beta, h) = e^{-(h/\theta)n} \mathbb{E} \left[ \prod_{\gamma \in \mathcal{C}} (e^{h|\gamma|} + \theta - 1) \right],$$

where  $|\gamma|$  denotes the size of the cycle  $\gamma$ . Note that  $Z_n(\beta, 0) = Z_n(\beta)$ . We will later (see Theorem 3.5) obtain an explicit expression for the limit  $z(\beta, h) = \lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\beta, h)$ . (Theorem 1.1 is the special case  $h = 0$  of that result.) Our second main result concerns the right derivative  $z^+(\beta) = \lim_{h \downarrow 0} \frac{z(\beta, h) - z(\beta, 0)}{h}$  of  $z(\beta, h)$  at  $h = 0$ .

**THEOREM 1.2.** *Define*

$$(9) \quad \beta_c(\theta) = \begin{cases} 2, & \text{if } \theta = 2, \\ 2\left(\frac{\theta-1}{\theta-2}\right) \log(\theta-1), & \text{if } \theta \geq 3. \end{cases}$$

*Then for all  $\theta \in \{2, 3, \dots\}$  we have that*

$$(10) \quad z^+(\beta) \begin{cases} = 0, & \text{if } \beta < \beta_c, \text{ or } \theta = 2 \text{ and } \beta = \beta_c, \\ > 0, & \text{if } \beta > \beta_c, \text{ or } \theta \geq 3 \text{ and } \beta = \beta_c. \end{cases}$$

Thus, the critical inverse-temperature is given by (9), and the phase-transition is continuous for  $\theta = 2$  (i.e.  $S = \frac{1}{2}$ ) and discontinuous for  $\theta \geq 3$  (i.e.  $S \geq 1$ ). We reiterate that the case  $\theta = 2$  was fully understood previously [19, 22].

**1.2. Discussion.** Theorem 1.2 has consequences for the following *weighted interchange process*. Recall the measure  $\mathbb{P}$  governing the random permutation  $\sigma(\omega)$ , obtained as the composition of a process of transpositions up to time  $\beta/n$ . For each  $\theta > 0$  one can define another probability measure  $\mathbb{P}_\theta$  by requiring  $\frac{d\mathbb{P}_\theta}{d\mathbb{P}} \propto \theta^{\ell(\omega)}$ . The measure  $\mathbb{P}_\theta$  allows for probabilistic interpretation of correlation functions. For example when  $S = \frac{1}{2}$ :

$$\langle S_x^3 S_y^3 \rangle = \frac{1}{2} \mathbb{P}_2(x \leftrightarrow y),$$

where  $\{x \leftrightarrow y\}$  is the event that  $x$  and  $y$  belong to the same cycle. Similar relations hold for other  $\theta$ . Magnetic ordering is thus accompanied by the occurrence of large cycles in a  $\mathbb{P}_\theta$ -distributed random permutation.

For each  $k \geq 0$  let  $X_n(k) = \frac{1}{n} \sum_{|\gamma| \geq k} |\gamma|$  denote the fraction of vertices in cycles of size at least  $k$  in the random permutation  $\sigma(\omega)$ . From Theorem 1.2 we will deduce the following:

**PROPOSITION 1.3.** *If  $\theta \in \{2, 3, \dots\}$  and  $z^+(\beta) = 0$  then for any sequence  $k = k_n \rightarrow \infty$  and any fixed  $\varepsilon > 0$ , there is a  $c > 0$  such that*

$$\mathbb{P}_\theta(X_n(k) \geq \varepsilon) \leq e^{-cn}.$$

Tóth's formula [23, eq. (5.2)] suggests that a converse to Proposition 1.3 should also hold, i.e. that there are cycles of size of the order  $n$  when  $z^+(\beta) > 0$ . We have not been able to prove this. Note, however, that cycles of order  $n$  do occur whenever  $\beta > \theta \geq 1$ . For  $\theta = 1$  this was proved by Schramm [21], and for  $\theta > 1$  it was proved in [8] using Schramm's result.

Theorem 1.2 also points to a connection to the classical Potts model. In that model, one considers random assignments  $\eta = (\eta_x : x \in V)$  of the values  $1, 2, \dots, q$  to the vertices  $x \in V$ , for some fixed  $q \in \{2, 3, \dots\}$ . Each such assignment receives probability proportional to

$$\exp\left(\frac{\beta}{n} \sum_{xy \in E} \delta_{\eta_x, \eta_y}\right).$$

It was proved by Bollobás, Grimmett and Janson in [12] (in the more general context of the random-cluster-representation) that a phase-transition occurs in this model at the point  $\beta = \beta_c(q)$  with  $\beta_c(\cdot)$  as given in (9). This equality of critical points may indicate a deeper connection between the two models, which we hope to explore in future work.

**1.3. Outline.** Over the following three sections we will prove somewhat more detailed versions of Theorems 1.1 and 1.2 and Proposition 1.3. In Section 2 we first obtain a formula for  $Z_n(\beta, h)$  for finite  $n$ , stated in Lemma 2.2. This formula is amenable to asymptotic analysis, which we perform in Section 3. The main result of that Section is Theorem 3.5, where we compute  $\lim_{n \rightarrow \infty} \frac{1}{n} \log Z_n(\beta, h)$ . In Section 4

we use the latter result to describe the phase transition and identify the critical point. Some additional proofs are given in the Appendix.

## 2. CHARACTER DECOMPOSITION OF THE PARTITION FUNCTION

In this section we obtain an expression for the partition function  $Z_n(\beta, h)$  in terms of the irreducible characters of the symmetric group. From now on we will usually only refer to the spin  $S \in \frac{1}{2}\mathbb{N}$  via the parameter  $\theta = 2S + 1 \in \{2, 3, \dots\}$ . Recall that  $\sigma = \sigma(\omega) \in \mathcal{S}_n$  is the random permutation introduced below (6), that  $\mathcal{C} = \mathcal{C}(\omega)$  is the set of cycles in a disjoint-cycle decomposition of  $\sigma$ , and that  $\ell = \ell(\omega) = |\mathcal{C}(\omega)|$  is the number of cycles.

By a *composition*  $\kappa$  of  $n$  we mean a vector  $\kappa = (\kappa_1, \dots, \kappa_\theta)$  with non-negative integer entries, such that  $\sum_{j=1}^\theta \kappa_j = n$ . Note that we restrict the number of entries to be exactly  $\theta$ , and that we allow some  $\kappa_j$  to be  $= 0$ . A composition  $\lambda$  is called a *partition* if in addition  $\lambda_j \geq \lambda_{j+1}$  for all  $j$ , in which case we write  $\lambda \vdash n$ . Any composition may be rearranged to form a partition. Given a partition  $\lambda$ , let  $K(\lambda)$  denote the set of compositions that can be obtained by re-ordering the entries of  $\lambda$ . Clearly  $1 \leq |K(\lambda)| \leq \theta!$ . We write  $\binom{n}{\lambda}$  for the multinomial coefficient

$$\binom{n}{\lambda} = \frac{n!}{\lambda_1! \lambda_2! \cdots \lambda_\theta!}.$$

**2.1. Colouring-lemma.** Let  $p_1, \dots, p_\theta$  be probabilities, i.e. non-negative numbers summing to 1. Write  $f(\sigma) = \mathbb{P}(\sigma(\omega) = \sigma)$  for the distribution function of  $\sigma(\omega)$ . Note that  $f(\cdot)$  is a class-function, i.e.  $f(\sigma) = f(\pi)$  whenever  $\sigma$  and  $\pi$  have the same cycle-type. (This uses that we are working on the complete graph.) For  $\lambda \vdash n$  we write  $\mathcal{T}_\lambda$  for the *Young subgroup* of  $\mathcal{S}_n$ , i.e. the subgroup consisting of those permutations which fix each of the sets

$$\{1, \dots, \lambda_1\}, \quad \{\lambda_1 + 1, \dots, \lambda_1 + \lambda_2\}, \quad \text{etc.}$$

LEMMA 2.1 (Colouring-lemma). *We have that*

$$\mathbb{E} \left[ \prod_{\gamma \in \mathcal{C}} \left( \sum_{i=1}^\theta p_i^{|\gamma|} \right) \right] = \sum_{\lambda \vdash n} \binom{n}{\lambda} \left( \sum_{\kappa \in K(\lambda)} \prod_{i=1}^\theta p_i^{\kappa_i} \right) \sum_{\sigma \in \mathcal{T}_\lambda} f(\sigma).$$

*Proof.* Colour each vertex of  $V = \{1, \dots, n\}$  independently using the colours  $1, \dots, \theta$ , colour  $\#i$  with probability  $p_i$ . Write  $\mathcal{M}$  for the event that all cycles of  $\sigma$  are monochromatic. The conditional probability of  $\mathcal{M}$  given  $\sigma$  is

$$\prod_{\gamma \in \mathcal{C}} \left( \sum_{i=1}^\theta p_i^{|\gamma|} \right),$$

so the left-hand-side of the claim is just  $\mathbb{P}(\mathcal{M})$ . On the other hand, by assigning the colours first we see that

$$(11) \quad \mathbb{P}(\mathcal{M}) = \sum_{C_1, \dots, C_\theta} \left( \prod_{i=1}^{\theta} p_i^{|C_i|} \right) \mathbb{P}(\sigma \in \mathcal{T}(C_1, \dots, C_\theta))$$

where the sum is over all (ordered) set partitions  $C_1, \dots, C_\theta$  of  $\{1, \dots, n\}$ , and  $\mathcal{T}(C_1, \dots, C_\theta)$  is the subgroup of  $\mathcal{S}_n$  consisting of permutations which fix each of the sets  $C_i$ .

Let  $\lambda \vdash n$  be the partition of  $n$  obtained by ordering the  $|C_i|$  by size. Then there is some  $\pi \in \mathcal{S}_n$  such that

$$(12) \quad \pi^{-1} \mathcal{T}(C_1, \dots, C_\theta) \pi = \mathcal{T}_\lambda.$$

Indeed, conjugation corresponds to relabelling the vertices, so we simply choose the appropriate relabelling of the sets  $C_i$ . It follows that

$$\begin{aligned} \mathbb{P}(\sigma \in \mathcal{T}(C_1, \dots, C_\theta)) &= \sum_{\sigma \in \mathcal{T}(C_1, \dots, C_\theta)} f(\sigma) = \sum_{\sigma \in \mathcal{T}_\lambda} f(\pi \sigma \pi^{-1}) \\ &= \sum_{\sigma \in \mathcal{T}_\lambda} f(\sigma), \end{aligned}$$

since  $f(\cdot)$  is a class-function. Putting this into (11) and summing over all possible  $\lambda \vdash n$  we get that

$$(13) \quad \mathbb{P}(\mathcal{M}) = \sum_{\lambda \vdash n} \left( \sum_{\sigma \in \mathcal{T}_\lambda} f(\sigma) \right) \left( \sum_{C_1, \dots, C_\theta} \prod_{i=1}^{\theta} p_i^{|C_i|} \right)$$

where now the sum over the  $C_i$  is restricted to those with the property that  $(|C_1|, \dots, |C_\theta|) \in K(\lambda)$ . This sum may be performed by first summing over all  $\kappa \in K(\lambda)$ , and then over all choices of the sets  $C_i$  with  $\kappa_i = |C_i|$ . For each fixed  $\kappa$ , there are  $\binom{n}{\lambda}$  choices of the sets. It follows that

$$\sum_{C_1, \dots, C_\theta} \prod_{i=1}^{\theta} p_i^{|C_i|} = \binom{n}{\lambda} \sum_{\kappa \in K(\lambda)} \prod_{i=1}^{\theta} p_i^{\kappa_i},$$

which proves the claim.  $\square$

Introduce the notation

$$(14) \quad G_n(\lambda) = \binom{n}{\lambda} \mathbb{P}(\sigma \in \mathcal{T}_\lambda) = \binom{n}{\lambda} \sum_{\sigma \in \mathcal{T}_\lambda} f(\sigma), \quad \text{for } \lambda \vdash n.$$

Taking all the  $p_i = \frac{1}{\theta}$  in Lemma 2.1 and using (7) we get (cancelling a factor  $\theta^{-n}$ ) that  $Z_n(\beta) = \mathbb{E}[\theta^{\ell(\omega)}] = \sum_{\lambda \vdash n} |K(\lambda)| G_n(\lambda)$ . More generally, we may take

$$(15) \quad p_1 = p e^h, \quad p_2 = \dots = p_\theta = p, \quad \text{for } h \in \mathbb{R},$$

with appropriate normalization  $p = (e^h + \theta - 1)^{-1}$ . Lemma 2.1 and (8) give that

$$(16) \quad e^{(h/\theta)n} Z_n(\beta, h) = \mathbb{E} \left[ \prod_{\gamma \in \mathcal{C}} (e^{h|\gamma|} + \theta - 1) \right] = \sum_{\lambda \vdash n} \left( \sum_{\kappa \in K(\lambda)} e^{h\kappa_1} \right) G_n(\lambda).$$

The factors  $\sum_{\kappa} e^{h\kappa_1}$  are bounded by simple expressions. Indeed, if  $h \geq 0$  then, since  $e^{h\lambda_1}$  is a summand in the sum over  $K(\lambda)$ , we have that

$$(17) \quad \sum_{\kappa \in K(\lambda)} e^{h\kappa_1} = e^{h\lambda_1} \sum_{\kappa \in K(\lambda)} e^{h(\kappa_1 - \lambda_1)} \begin{cases} \geq e^{h\lambda_1} \\ \leq \theta! e^{h\lambda_1}, \end{cases}$$

since  $\lambda_1$  is the largest of the  $\kappa_i$ . Similarly, if  $h \leq 0$  then

$$(18) \quad \sum_{\kappa \in K(\lambda)} e^{h\kappa_1} \begin{cases} \geq e^{h\lambda_\theta} \\ \leq \theta! e^{h\lambda_\theta}. \end{cases}$$

We will use the notation  $f(n) \asymp g(n)$  to denote that there is a constant  $C > 0$  such that  $\frac{1}{C}g(n) \leq f(n) \leq Cg(n)$  for all  $n$ . We may summarize the above as follows:

LEMMA 2.2. *With  $G_n(\lambda) = \binom{n}{\lambda} \mathbb{P}(\sigma \in \mathcal{T}_\lambda)$  as in (14), we have that*

$$Z_n(\beta) \asymp \sum_{\lambda \vdash n} G_n(\lambda), \quad \text{and} \\ e^{(h/\theta)n} Z_n(\beta, h) \asymp \begin{cases} \sum_{\lambda \vdash n} e^{h\lambda_1} G_n(\lambda), & \text{if } h > 0, \\ \sum_{\lambda \vdash n} e^{h\lambda_\theta} G_n(\lambda), & \text{if } h < 0. \end{cases}$$

**2.2. Some representation theory.** From Lemma 2.2 it is clear that the probabilities  $\mathbb{P}(\sigma \in \mathcal{T}_\lambda)$  are important. We now express them using the irreducible representations of  $\mathcal{S}_n$ . For background on the representation theory of  $\mathcal{S}_n$  we refer to e.g. Fulton–Harris [15].

The irreducible representations of  $\mathcal{S}_n$  are indexed by partitions  $\mu \vdash n$ . (In this description we temporarily omit our convention that partitions have at most  $\theta$  non-zero parts.) It is convenient to represent  $\mu \vdash n$  by its Young-diagram, as in Figure 2. We write  $U_\mu$  for the irreducible representation corresponding to  $\mu \vdash n$ , and  $\chi_\mu$  for its character. Let  $V_\lambda$  denote the coset representation of the subgroup  $\mathcal{T}_\lambda$ , that is  $V_\lambda$  is a vector space spanned by the cosets  $\pi\mathcal{T}_\lambda$  and  $\mathcal{S}_n$  acts by left multiplication. By Young’s rule [15, Corollary 4.39], the representation  $V_\lambda$  decomposes as a direct sum of irreducible representations with known multiplicities:

$$(19) \quad V_\lambda = \bigoplus_{\mu \vdash n} K_{\mu\lambda} U_\mu.$$

Here the multiplicities  $K_{\mu\lambda}$  are the *Kostka numbers*:  $K_{\mu\lambda}$  equals the number of ways to fill the Young diagram for  $\mu$  with  $\lambda_1$  1’s,  $\lambda_2$  2’s et.c. so that the rows are weakly increasing and the columns are strictly increasing. See Figure 2 again.



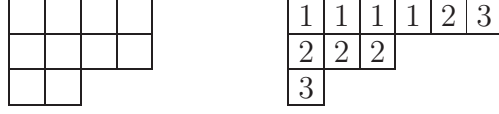


FIGURE 2. Left: Young diagram of the partition  $\lambda = (4, 4, 2) \vdash 10$ . Right: diagram for  $\mu = (6, 3, 1) \supseteq \lambda$  filled with  $\lambda_1 = 4$  1's,  $\lambda_2 = 4$  2's and  $\lambda_3 = 2$  3's so that rows are weakly increasing and columns strictly increasing.

We say that  $\mu$  *dominates*  $\lambda$ , written  $\mu \supseteq \lambda$ , if for each  $i$  we have that  $\mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i$ . Note that  $K_{\mu\lambda} = 0$  unless  $\mu \supseteq \lambda$ . In particular, if  $\lambda$  has at most  $\theta$  non-zero parts, then  $K_{\mu\lambda} = 0$  unless  $\mu$  also has at most  $\theta$  non-zero parts. Writing  $\psi_\lambda$  for the character of  $V_\lambda$  it follows from (19) that

$$(20) \quad \psi_\lambda = \sum_{\mu \vdash n} K_{\mu\lambda} \chi_\mu.$$

Let  $\langle \cdot, \cdot \rangle$  denote the inner product of functions on  $\mathcal{S}_n$  given by

$$\langle f, g \rangle = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f(\sigma) \overline{g(\sigma)}.$$

Lemma 1 in Alon–Kozma [2] tells us that for a class function  $f$  we have

$$(21) \quad \sum_{\sigma \in \mathcal{T}_\lambda} f(\sigma) = |\mathcal{T}_\lambda| \langle f, \psi_\lambda \rangle$$

so using (20) we see that

$$(22) \quad \sum_{\sigma \in \mathcal{T}_\lambda} f(\sigma) = |\mathcal{T}_\lambda| \sum_{\mu \vdash n} K_{\mu\lambda} \langle f, \chi_\mu \rangle.$$

Now let  $f(\sigma) = \mathbb{P}(\sigma(\omega) = \sigma)$  as before. As already noted, this is a class-function. The calculations in Lemma 1 of Berestycki–Kozma [5] show that

$$(23) \quad \langle f, \chi_\mu \rangle = \frac{1}{n!} \text{tr}(\hat{f}(\mu)) = \frac{1}{n!} d_\mu \exp \left\{ \frac{\beta}{n} \binom{n}{2} [r(\mu) - 1] \right\}.$$

Here  $\hat{f}(\mu)$  denotes the Fourier transform of  $f$  at the irreducible representation  $U_\mu$ , the number  $d_\mu$  is the dimension of  $U_\mu$ , and finally  $r(\mu) = \chi_\mu((1, 2))/d_\mu$  is the character ratio at a transposition. We note for future reference that

$$(24) \quad \frac{\beta}{n} \binom{n}{2} [r(\mu) - 1] = \frac{\beta}{2} \left[ \sum_{j=1}^{\theta} \frac{\mu_j(\mu_j - 2j + 1)}{n} - (n - 1) \right]$$

see e.g. equation (7) in [5].

Putting together (22) and (23) gives

$$(25) \quad \mathbb{P}(\sigma \in \mathcal{T}_\lambda) = \sum_{\sigma \in \mathcal{T}_\lambda} f(\sigma) = \frac{|T_\lambda|}{n!} \sum_{\mu \vdash n} d_\mu K_{\mu\lambda} \exp \left\{ t \binom{n}{2} [r(\mu) - 1] \right\}.$$

Noting that  $\frac{|T_\lambda|}{n!} = \binom{n}{\lambda}^{-1}$ , we obtain:

LEMMA 2.3.

$$G_n(\lambda) = \sum_{\mu \vdash n} d_\mu K_{\mu\lambda} \exp \left\{ \frac{\beta}{n} \binom{n}{2} [r(\mu) - 1] \right\}.$$

For  $\theta = 2$  the partitions  $\mu$  can be indexed by the length of the second row, and it is well-known (and easy to see) that  $K_{\mu\lambda} = 1$  when  $\mu \supseteq \lambda$ . In that case Lemma 2.3 is essentially [19, eq. (49)].

### 3. CONVERGENCE-RESULTS

In this section will use the expressions in Lemmas 2.2 and 2.3 to identify the limit of  $\frac{1}{n} \log Z_n(\beta, h)$ .

**3.1. Lemmas.** We first present convergence-results in a slightly more general form, which we will later apply to our specific problem. Some of the arguments in this subsection are strongly inspired by [19, Section 6] and [20, Section 3.4].

Recall that

$$\Delta = \{(x_1, \dots, x_\theta) \in [0, 1]^\theta : x_1 \geq \dots \geq x_\theta, \sum x_i = 1\}.$$

For  $x, y \in \Delta$  we write  $y \supseteq x$  if  $y_1 + \dots + y_i \geq x_1 + \dots + x_i$  for all  $i$ . For  $x \in \Delta$  we write

$$(26) \quad \Delta(x) = \{y \in \Delta : y \supseteq x\}.$$

It is not hard to see that  $\Delta(\frac{1}{\theta}, \dots, \frac{1}{\theta}) = \Delta$ . Also note that each  $\Delta(x)$ , and hence also  $\Delta$ , is compact and convex.

Write  $\|\cdot\|$  for the  $\infty$ -norm on  $\mathbb{R}^\theta$ ,  $\|x - y\| = \max_{i=1, \dots, \theta} |x_i - y_i|$ . Write  $d_H(\cdot, \cdot)$  for the associated Hausdorff distance between sets in  $\mathbb{R}^\theta$ :

$$d_H(A, B) = \inf\{\varepsilon \geq 0 : A \subseteq B^\varepsilon \text{ and } B \subseteq A^\varepsilon\}$$

where  $A^\varepsilon = \{x \in \mathbb{R}^\theta : \|x - a\| < \varepsilon \text{ for some } a \in A\}$ .

The proof of the following result is given in Appendix B.

LEMMA 3.1. *Let  $x, y \in \Delta$  with  $\|x - y\| \leq \varepsilon < \theta^{-2}$ . Then*

$$d_H(\Delta(x), \Delta(y)) < \theta \varepsilon^{1/2}.$$

Now let  $\phi : \Delta \rightarrow \mathbb{R}$  be any continuous function (we will later take  $\phi = \phi_\beta$ ). Since  $\Delta$  is compact,  $\phi$  is uniformly continuous. Let  $\phi_n^{(\lambda)}(\mu)$  be a sequence of functions of partitions  $\lambda, \mu \vdash n$  converging *uniformly* to  $\phi$  in the following sense: there is a sequence  $\delta_n \rightarrow 0$ , not depending on  $\lambda$  or  $\mu$ , such that  $|\phi_n^{(\lambda)}(\mu) - \phi(\mu/n)| \leq \delta_n$  for all  $n$ .

LEMMA 3.2. *If  $n \rightarrow \infty$  and  $\lambda/n \rightarrow x \in \Delta$  then*

$$\frac{1}{n} \log \left( \sum_{\mu \succeq \lambda} \exp \left( n \phi_n^{(\lambda)}(\mu) \right) \right) \rightarrow \max_{y \in \Delta(x)} \phi(y).$$

*The maximum is attained since  $\Delta(x)$  is compact and  $\phi$  continuous.*

*Proof.* We first prove an upper bound. Since the number of partitions of  $n$  into at most  $\theta$  parts is at most  $n^\theta$  we have that

$$\begin{aligned} \sum_{\mu \succeq \lambda} \exp \left( n \phi_n^{(\lambda)}(\mu) \right) &\leq n^\theta \max_{\mu \succeq \lambda} \exp \left( n \phi_n^{(\lambda)}(\mu) \right) \\ &\leq n^\theta \exp \left( n \max_{\mu \succeq \lambda} \phi(\mu/n) + n\delta_n \right) \end{aligned}$$

so that

$$\frac{1}{n} \log \left( \sum_{\mu \succeq \lambda} \exp \left( n \phi_n^{(\lambda)}(\mu) \right) \right) \leq o(1) + \max_{\mu \succeq \lambda} \phi(\mu/n).$$

Let  $x_n = \lambda/n$ , then  $\mu \succeq \lambda$  is equivalent to  $\mu/n \in \Delta(x_n)$ , so we have that

$$\max_{\mu \succeq \lambda} \phi(\mu/n) \leq \max_{y \in \Delta(x_n)} \phi(y) = \phi(y_n^*)$$

for some  $y_n^* \in \Delta(x_n)$ . Now we use Lemma 3.1: given any  $\delta > 0$  we have, for  $n$  large enough, that there is some  $x_n^* \in \Delta(x)$  such that  $\|x_n^* - y_n^*\| < \delta$ . Since  $\phi$  is uniformly continuous we may, given  $\varepsilon > 0$ , pick  $\delta$  so that  $\|x_n^* - y_n^*\| < \delta$  implies  $|\phi(x_n^*) - \phi(y_n^*)| < \varepsilon$ . Then

$$\phi(y_n^*) \leq \phi(x_n^*) + \varepsilon \leq \max_{y \in \Delta(x)} \phi(y) + \varepsilon,$$

since  $x_n^* \in \Delta(x)$ . This shows that

$$\frac{1}{n} \log \left( \sum_{\mu \succeq \lambda} \exp \left( n \phi_n^{(\lambda)}(\mu) \right) \right) \leq o(1) + \max_{y \in \Delta(x)} \phi(y) + \varepsilon,$$

for any  $\varepsilon > 0$ , so

$$\limsup_{n \rightarrow \infty, \lambda/n \rightarrow x} \frac{1}{n} \log \left( \sum_{\mu \succeq \lambda} \exp \left( n \phi_n^{(\lambda)}(\mu) \right) \right) \leq \max_{y \in \Delta(x)} \phi(y).$$

Now for the lower bound. Pick some  $x^* \in \Delta(x)$  where  $\phi$  attains its maximum over  $\Delta(x)$ . As before, write  $x_n = \lambda/n$ . Using Lemma 3.1 as before, given  $\delta > 0$  we have that  $\Delta(x_n)$  intersects the ball  $B_\delta(x^*)$  of radius  $\delta$  around  $x^*$  provided that  $n$  is large enough. Write  $\overline{B}_\delta(x^*)$  for the closed ball. By the triangle inequality we may further assume that

$\Delta(x_n) \cap \overline{B}_\delta(x^*)$  contains some point of the form  $\mu/n$ . Thus

$$\begin{aligned}
 \sum_{\mu \geq \lambda} \exp(n \phi_n^{(\lambda)}(\mu)) &\geq \sum_{\mu/n \in \Delta(x_n)} \exp(n \phi(\mu/n) - n\delta_n) \\
 &\geq \min_{\mu/n \in \Delta(x_n) \cap \overline{B}_\delta(x^*)} \exp(n \phi(\mu/n) - n\delta_n) \\
 (27) \quad &\geq \min_{y \in \Delta \cap \overline{B}_\delta(x^*)} \exp(n \phi(y) - n\delta_n) \\
 &= \exp\left(n \min_{y \in \Delta \cap \overline{B}_\delta(x^*)} \phi(y) - n\delta_n\right).
 \end{aligned}$$

Hence

$$\frac{1}{n} \log \left( \sum_{\mu \geq \lambda} \exp(n \phi_n^{(\lambda)}(\mu)) \right) \geq \min_{y \in \Delta \cap \overline{B}_\delta(x^*)} \phi(y) - \delta_n.$$

By the uniform continuity of  $\phi$ , given  $\varepsilon > 0$  we may pick  $\delta$  small enough such that

$$\min_{y \in \Delta \cap \overline{B}_\delta(x^*)} \phi(y) \geq \phi(x^*) - \varepsilon.$$

This gives

$$\limsup_{n \rightarrow \infty, \lambda/n \rightarrow x} \frac{1}{n} \log \left( \sum_{\mu \geq \lambda} \exp(n \phi_n^{(\lambda)}(\mu)) \right) \geq \phi(x^*) - \varepsilon,$$

which proves the claim.  $\square$

The next result may be established straightforwardly using Lemma 3.1:

**LEMMA 3.3.** *The function  $g : \Delta \rightarrow \mathbb{R}$  given by  $g(x) = \max_{y \geq x} \phi(y)$  is continuous.*

We next present a slight extension of Lemma 3.2. We assume that  $\phi$  and  $\phi_n^{(\lambda)}(\mu)$  are as before. Write  $y = (y_1, \dots, y_\theta) \in \mathbb{R}^\theta$  and  $y \cdot x = \sum y_i x_i$  for the usual product.

**LEMMA 3.4.** *For any  $y \in \mathbb{R}^\theta$  we have as  $n \rightarrow \infty$  that*

$$\frac{1}{n} \log \left( \sum_{\lambda \vdash n} e^{y \cdot \lambda} \sum_{\mu \geq \lambda} \exp(n \phi_n^{(\lambda)}(\mu)) \right) \rightarrow \max_{x \in \Delta} (y \cdot x + g(x))$$

where  $g$  is the function in Lemma 3.3.

*Proof.* Write  $m(y) = \max_{x \in \Delta} (y \cdot x + g(x))$ . Bounding the number of partitions by  $n^\theta$  as before, we see that

$$\begin{aligned} \sum_{\lambda \vdash n} e^{y \cdot \lambda} \sum_{\mu \succeq \lambda} \exp(n \phi_n^{(\lambda)}(\mu)) &\leq n^{2\theta} \cdot \max_{\lambda \vdash n} \max_{\mu \succeq \lambda} \exp(y \cdot \lambda + n \phi(\mu/n) + n\delta_n) \\ &\leq n^{2\theta} \cdot \max_{\lambda \vdash n} \max_{z \in \Delta(\lambda/n)} \exp(y \cdot \lambda + n \phi(z) + n\delta_n) \\ &\leq n^{2\theta} \cdot \max_{x \in \Delta} \max_{z \in \Delta(x)} \exp(ny \cdot x + n \phi(z) + n\delta_n) \\ &\leq n^{2\theta} \cdot \exp\left(\max_{x \in \Delta} \{ny \cdot x + ng(x)\} + n\delta_n\right) \end{aligned}$$

Thus

$$\frac{1}{n} \log \left( \sum_{\lambda \vdash n} e^{y \cdot \lambda} \sum_{\mu \succeq \lambda} \exp(n \phi_n^{(\lambda)}(\mu)) \right) \leq m(y) + o(1).$$

For the lower bound, note that given  $\delta > 0$  we may find  $\tilde{x} \in \Delta$  such that

$$y \cdot \tilde{x} + g(\tilde{x}) \geq m(y) - \delta.$$

We may also find a sequence  $\tilde{\lambda} \vdash n$  such that  $\tilde{\lambda}/n \rightarrow \tilde{x}$ . Clearly

$$\sum_{\lambda \vdash n} e^{y \cdot \lambda} \sum_{\mu \succeq \lambda} \exp(n \phi_n^{(\lambda)}(\mu)) \geq e^{y \cdot \tilde{\lambda}} \sum_{\mu \succeq \tilde{\lambda}} \exp(n \phi_n^{(\tilde{\lambda})}(\mu))$$

and hence

$$\frac{1}{n} \log \left( \sum_{\lambda \vdash n} e^{y \cdot \lambda} \sum_{\mu \succeq \lambda} \exp(n \phi_n^{(\lambda)}(\mu)) \right) \geq \frac{y \cdot \tilde{\lambda}}{n} + \frac{1}{n} \log \left( \sum_{\mu \succeq \tilde{\lambda}} \exp(n \phi_n^{(\tilde{\lambda})}(\mu)) \right).$$

By Lemma 3.2 the right-hand-side converges to

$$y \cdot \tilde{x} + g(\tilde{x}) \geq m(y) - \delta.$$

This proves the claim.  $\square$

**3.2. The free energy.** From now on we let  $\phi = \phi_\beta : \Delta \rightarrow \mathbb{R}$  be the function given in (4), i.e.  $\phi_\beta(x) = \frac{\beta}{2} \left( \sum_{i=1}^\theta x_i^2 - 1 \right) - \sum_{i=1}^\theta x_i \log x_i$ . Note that  $\phi_\beta$  is continuous. We write  $g_\beta(x) = \max_{y \succeq x} \phi_\beta(y)$  and we define

$$(28) \quad z(\beta, h) = \begin{cases} \max_{x \in \Delta} \left( h(x_1 - \frac{1}{\theta}) + g_\beta(x) \right), & \text{if } h \geq 0, \\ \max_{x \in \Delta} \left( h(x_\theta - \frac{1}{\theta}) + g_\beta(x) \right), & \text{if } h \leq 0. \end{cases}$$

Note that  $x_1 - \frac{1}{\theta} \geq 0$  and  $x_\theta - \frac{1}{\theta} \leq 0$ .

The following theorem contains Theorem 1.1 as the case  $h = 0$ .

**THEOREM 3.5.** *We have that*

$$\frac{1}{n} \log G_n(\lambda) \rightarrow g_\beta(x), \text{ as } n \rightarrow \infty \text{ and } \lambda/n \rightarrow x,$$

and for  $h \in \mathbb{R}$  that

$$\frac{1}{n} \log Z_n(\beta, h) \rightarrow z(\beta, h), \text{ as } n \rightarrow \infty.$$

*Proof.* We will use Lemmas 3.2 and 3.4 with

$$(29) \quad \phi_n^{(\lambda)}(\mu) = \frac{\beta}{n^2} \binom{n}{2} [r(\mu) - 1] + \frac{1}{n} \log d_\mu + \frac{1}{n} \log K_{\mu\lambda}.$$

Due to Lemmas 2.2 and 2.3 it suffices to establish the uniform convergence of  $\phi_n^{(\lambda)}(\mu)$  to  $\phi = \phi_\beta$ . First note that  $K_{\mu\lambda} \leq (n+1)^{\theta^2}$ . Indeed, for each row of  $\mu$  we must choose the number of 1's, the number of 2's etc. Thus there are certainly at most  $(\lambda_1 + 1) \cdots (\lambda_\theta + 1)$  choices for each row, and thus

$$K_{\mu\lambda} \leq [(\lambda_1 + 1) \cdots (\lambda_\theta + 1)]^\theta \leq (n+1)^{\theta^2},$$

as claimed. Defining

$$\phi_n(\mu) = \frac{\beta}{n^2} \binom{n}{2} [r(\mu) - 1] + \frac{1}{n} \log d_\mu$$

we thus have that

$$|\phi_n^{(\lambda)}(\mu) - \phi(\mu/n)| \leq |\phi_n(\mu) - \phi(\mu/n)| + \frac{\theta^2}{n} \log(n+1).$$

Now by (24) we have

$$\frac{\beta}{n^2} \binom{n}{2} [r(\mu) - 1] = \frac{\beta}{2} \left[ \sum_{j=1}^{\theta} \frac{\mu_j(\mu_j - 2j + 1)}{n^2} - \frac{n-1}{n} \right]$$

and we have that

$$\left| \sum_{j=1}^{\theta} \frac{\mu_j(\mu_j - 2j + 1)}{n^2} - \sum_{j=1}^{\theta} \left( \frac{\mu_j}{n} \right)^2 \right| \leq \sum_{j=1}^{\theta} \frac{\mu_j}{n} \left( \frac{2j-1}{n} \right) \leq \frac{2\theta-1}{n}.$$

Next, (4.11) on page 50 of [15] gives that

$$\log d_\mu = \log \left( \frac{n!}{m_1! \cdots m_k!} \prod_{1 \leq i < j \leq k} (m_i - m_j) \right)$$

where  $m_i = \mu_i + k - i$  and  $k$  is the number of *nonzero* parts of  $\mu$ . Thus

$$\begin{aligned} \left| \frac{1}{n} \log d_\mu - \frac{1}{n} \log \binom{n}{\mu} \right| &\leq \frac{1}{n} \log \prod_{1 \leq i < j \leq k} (m_i - m_j) \\ &+ \frac{1}{n} \log [(\mu_1 + k - 1) \cdots (\mu_1 + 1)(\mu_2 + k - 2) \cdots (\mu_2 + 1) \cdots (\mu_{k-1} + 1)] \\ &\leq \frac{1}{n} \log(n + \theta - 1)^{\theta^2} + \frac{1}{n} \log(n + \theta - 1)^\theta. \end{aligned}$$

Thus it suffices to bound

$$\left| \frac{1}{n} \log \binom{n}{\mu} - \left( - \sum_{j=1}^{\theta} \frac{\mu_j}{n} \log \frac{\mu_j}{n} \right) \right|.$$

But by Stirling's formula

$$\binom{n}{\mu} \asymp \left( \frac{n}{\prod_{j=1}^{\theta} \mu_j} \right)^{1/2} \prod_{j=1}^{\theta} \left( \frac{n}{\mu_j} \right)^{\mu_j}$$

so that

$$\left| \frac{1}{n} \log \binom{n}{\mu} - \left( - \sum_{j=1}^{\theta} \frac{\mu_j}{n} \log \frac{\mu_j}{n} \right) \right| \leq \left| \frac{1}{n} \log \left( \frac{n}{\prod_{j=1}^{\theta} \mu_j} \right)^{1/2} \right| + \frac{C}{n}.$$

The right-hand-side is at most  $\frac{C'}{n} \log n$ . This proves the result.  $\square$

#### 4. THE PHASE-TRANSITION

In this last section we prove Theorem 1.2 and Proposition 1.3. Recall  $\phi_{\beta}$  and  $z(\beta, h)$  defined in (4) and (28), respectively.

**4.1. Left and right derivatives of  $z(\beta, h)$  at  $h = 0$ .** Let  $x^{\uparrow}(\beta) \in \Delta$  denote a maximizer of  $\phi_{\beta}$  which maximizes the first coordinate. That is, among the maximizers  $x$  of  $\phi_{\beta}$  we pick one for which  $x_1$  is maximal. Similarly, let  $x^{\downarrow}(\beta)$  denote a maximizer of  $\phi_{\beta}$  which *minimizes* the last coordinate  $x_{\theta}$ . Note that  $x^{\uparrow}$  and  $x^{\downarrow}$  depend on  $\beta$ , though we do not always write this explicitly.

The left and right derivatives of  $z(\beta, h)$  at  $h = 0$  are given by

$$z^+(\beta) = \lim_{h \downarrow 0} \frac{z(\beta, h) - z(\beta, 0)}{h}, \quad z^-(\beta) = \lim_{h \uparrow 0} \frac{z(\beta, h) - z(0)}{h}.$$

We will show:

**THEOREM 4.1.**

$$z^+(\beta) = x_1^{\uparrow}(\beta) - \frac{1}{\theta}, \quad z^-(\beta) = x_{\theta}^{\downarrow}(\beta) - \frac{1}{\theta}.$$

*Proof.* We prove the claim about  $z^+(\beta)$ ; the argument for  $z^-(\beta)$  is similar. First note that  $z(\beta, 0) = \phi_{\beta}(x^{\uparrow})$  and so

$$\frac{z(\beta, h) - z(\beta, 0)}{h} = \max_{x \in \Delta} f(x, h),$$

where

$$f(x, h) = x_1 - \frac{1}{\theta} + \frac{g_{\beta}(x) - \phi_{\beta}(x^{\uparrow})}{h}.$$

We have that  $f(x^{\uparrow}, h) = x_1^{\uparrow} - \frac{1}{\theta}$ , since  $g_{\beta}(x^{\uparrow}) = \phi_{\beta}(x^{\uparrow})$ , and thus

$$\frac{z(\beta, h) - z(\beta, 0)}{h} \geq x_1^{\uparrow} - \frac{1}{\theta} \text{ for all } h > 0.$$

Also,  $f$  is continuous as a function on  $\Delta \times (0, \infty)$ , thus for each  $h$  it attains its maximum at some point  $x(h) \in \Delta$ . Since  $g_{\beta}(x) \leq \phi_{\beta}(x^{\uparrow})$  for all  $x \in \Delta$  it follows that

$$x_1^{\uparrow} - \frac{1}{\theta} \leq f(x(h), h) \leq x_1(h) - \frac{1}{\theta}, \text{ for all } h > 0.$$

It thus suffices to show that  $x_1(h) \rightarrow x_1^\uparrow$  as  $h \downarrow 0$ .

If not then there is some  $\varepsilon > 0$  and some sequence  $h_i \downarrow 0$  such that  $x(h_i) \in A_\varepsilon$  for all  $i$ , where

$$A_\varepsilon = \{x \in \Delta : x_1 \geq x_1^\uparrow + \varepsilon\}.$$

Note that there is some  $\delta > 0$  such that  $\phi_\beta(x) \leq \phi_\beta(x^\uparrow) - \delta$  for all  $x \in A_\varepsilon$ , since  $\phi_\beta$  is continuous and  $A_\varepsilon$  compact. Also note that if  $x \in A_\varepsilon$  then  $\Delta(x) \subseteq A_\varepsilon$ , by the definition of  $\triangleright$ . Thus  $g_\beta(x(h_i)) \leq \phi_\beta(x^\uparrow) - \delta$  for all  $i$ . But then

$$f(x(h_i), h_i) = x_1(h_i) - \frac{1}{\theta} + \frac{g_\beta(x(h_i)) - \phi_\beta(x^\uparrow)}{h_i} \leq 1 - \frac{1}{\theta} - \frac{\delta}{h_i} \rightarrow -\infty.$$

This contradicts the fact that  $f(x, h) \geq x_1^\uparrow - \frac{1}{\theta}$  for all  $x \in \Delta$ . Hence it must be the case that  $x_1(h) \rightarrow x_1^\uparrow$ , as claimed.  $\square$

**4.2. The critical point.** In light of Theorem 4.1, the following result implies Theorem 1.2. Recall that  $\beta_c(\theta) := 2\left(\frac{\theta-1}{\theta-2}\right) \log(\theta-1)$  for  $\theta \geq 3$  and  $\beta_c(2) = 2$ .

**THEOREM 4.2.**

If  $\beta < \beta_c$ , or  $\theta = 2$  and  $\beta = \beta_c$ , then  $x_1^\uparrow = x_\theta^\downarrow = \frac{1}{\theta}$ .

If  $\beta > \beta_c$ , or  $\theta \geq 3$  and  $\beta = \beta_c$ , then  $x_1^\uparrow > \frac{1}{\theta}$  and  $x_\theta^\downarrow < \frac{1}{\theta}$ .

*Proof.* Since  $x_1 \geq \frac{1}{\theta}$  and  $x_\theta \leq \frac{1}{\theta}$  for all  $x \in \Delta$  we must determine when  $\phi_\beta$  has a maximizer different from  $(\frac{1}{\theta}, \dots, \frac{1}{\theta})$ . We start by characterizing the possible maxima of  $\phi_\beta$  using the Lagrangian necessity theorem. Since the functions  $\phi_\beta(x)$  and  $c(x) = \sum x_i - 1$  are  $C^1$  on  $(0, \infty)^\theta$ , and  $\nabla c(x)$  is nonzero for all  $x$ , if  $x \in \Delta$  is any local extremum of  $\phi_\beta$  then there is some  $a \in \mathbb{R}$  such that

$$\nabla \phi_\beta(x) = a \nabla c(x) = (a, \dots, a).$$

Now

$$(30) \quad \frac{\partial \phi_\beta}{\partial x_i} = \beta x_i - \log x_i - 1$$

so if  $x \in \Delta$  is a local maximum then there is some  $a \in \mathbb{R}$  such that

$$(31) \quad \beta x_i = (1 - a) + \log x_i, \quad \text{for all } i = 1, \dots, \theta.$$

(We see from (30) that the partial derivative diverges to  $+\infty$  if  $x_i \downarrow 0$ , thus  $\phi_\beta$  is not maximized on the boundary and it suffices to consider local maxima.) For each  $\beta > 0$  and  $a \in \mathbb{R}$ , there are 0, 1 or 2 values of  $x_i$  which satisfy (31). If there is just 1 solution then all the  $x_i$  are equal, and hence equal to  $\frac{1}{\theta}$ . If there are 2 solutions then, since  $\phi_\beta$  is symmetric in its arguments, we can assume that there is some  $1 \leq r \leq \theta - 1$  such that  $x$  is of the form

$$(32) \quad x = (t, \dots, t, \frac{1-rt}{\theta-r}, \frac{1-rt}{\theta-r}) \quad \text{for } \frac{1}{\theta} < t < \frac{1}{r},$$



with the first  $r$  coordinates equal and the last  $\theta - r$  coordinates equal. Write  $\phi_\beta^{(r)}(t)$  for  $\phi_\beta$  evaluated at  $x$  of the form (32).

We now establish a condition on  $\beta$  for  $\phi_\beta^{(r)}(t)$  to exceed  $\phi_\beta^{(r)}(\frac{1}{\theta})$  for some  $t > \frac{1}{\theta}$ . A short calculation shows that

$$\phi_\beta^{(r)}(t) - \phi_\beta^{(r)}(\frac{1}{\theta}) = \frac{\beta r}{2\theta(\theta - r)}(\theta t - 1)^2 - [rt \log t + (1 - rt) \log \frac{1 - rt}{\theta - r} + \log \theta].$$

Thus  $\phi_\beta^{(r)}(t) - \phi_\beta^{(r)}(\frac{1}{\theta}) \geq 0$  if and only if

$$(33) \quad \beta \geq R(t) = \left( \frac{2\theta(\theta - r)}{r} \right) \frac{rt \log t + (1 - rt) \log \frac{1 - rt}{\theta - r} + \log \theta}{(\theta t - 1)^2}.$$

Hence  $\phi_\beta$  has a maximizer different from  $(\frac{1}{\theta}, \dots, \frac{1}{\theta})$  if and only if  $\beta \geq R(t)$  for some  $r$  and some  $t > \frac{1}{\theta}$ . We show in Appendix C that  $R$  is convex. Also, note that  $R(t) \rightarrow +\infty$  as  $t \uparrow \frac{1}{r}$  and that  $R'(\frac{\theta - r}{r\theta}) = 0$ . Thus  $R(t)$  has a unique minimum in  $[\frac{1}{\theta}, \frac{1}{r})$ , either at the boundary point  $t = \frac{1}{\theta}$  if  $r > \theta/2$ , or at  $t = \frac{\theta - r}{r\theta}$  if  $r \leq \theta/2$ .

In the case when  $\theta = 2$  the only possibility for  $t > \frac{1}{\theta}$  is when  $r = 1$ . Then  $\frac{1}{\theta} = \frac{\theta - r}{r\theta} = \frac{1}{2}$  and hence  $\beta_c(2) = \inf_{t > 1/2} R(t) = 2$ . If  $\theta \geq 3$ , note that

$$(34) \quad R(\frac{\theta - r}{r\theta}) = \rho(\frac{r}{\theta}), \text{ with } \rho(t) = 2\theta t \frac{1 - t}{1 - 2t} \log \left( \frac{1 - t}{t} \right).$$

The function  $\rho$  is increasing on  $[0, \frac{1}{2}]$ , so  $\rho(\frac{r}{\theta})$  is minimal for  $r = 1$ . This gives the critical value  $\beta_c = \rho(\frac{1}{\theta})$  claimed.

To check the statements about  $x^\uparrow$  and  $x^\downarrow$  at  $\beta = \beta_c$ , we note that for this value of  $\beta$  we have a maximizer of  $\phi_\beta$  at the point (32) with  $r = 1$  and  $t = \frac{\theta - 1}{\theta}$ . Thus, at  $\beta = \beta_c$ ,

$$x_1^\uparrow = \frac{\theta - 1}{\theta} \quad \text{and} \quad x_\theta^\downarrow = \frac{1}{(\theta - 1)\theta}.$$

The claims follow.  $\square$

**4.3. The number of vertices in large cycles.** Let  $k = k_n \rightarrow \infty$  be any sequence going to  $\infty$ . Recall that  $X_n(k) = \frac{1}{n} \sum_{|\gamma| \geq k} |\gamma|$  denotes the fraction of vertices in cycles of size at least  $k$  in the random permutation  $\sigma(\omega)$ . We now show that, under  $\mathbb{P}_\theta$  with  $\theta \in \{2, 3, \dots\}$ , asymptotically  $X_n(k)$  is at most

$$(35) \quad \frac{\theta x_1^\uparrow - 1}{\theta - 1}.$$

Note that this number is  $= 0$  if and only if  $x_1^\uparrow = \frac{1}{\theta}$ , i.e.  $z^+(\beta) = 0$ . Proposition 1.3 is a special case of the following result:

**PROPOSITION 4.3.** *Let  $\beta > 0$ . For any  $\alpha < 1 - \frac{1}{\theta}$  and any  $\varepsilon > 0$  there is some  $c > 0$  such that*

$$(36) \quad \mathbb{P}_\theta(X_n(k) > \varepsilon + \frac{1}{\alpha}(x_1^\uparrow - \frac{1}{\theta})) \leq e^{-cn}$$

for all large enough  $n$ .

*Proof.* We claim that for any  $h > 0$  we have for large enough  $n$  that

$$(37) \quad \mathbb{E}_\theta \left[ \exp \left( \alpha h \sum_{|\gamma| \geq k} |\gamma| \right) \right] \leq \frac{Z_n(\beta, h)}{Z_n(\beta, 0)}.$$

Indeed,

$$(38) \quad \begin{aligned} Z_n(\beta, h) &= \mathbb{E} \left[ \prod_{\gamma} \left( \frac{e^{h|\gamma|} + \theta - 1}{e^{(h/\theta)|\gamma|}} \right) \right] \\ &= \mathbb{E} \left[ \theta^\ell \prod_{|\gamma| \geq k} w(h|\gamma|) \prod_{|\gamma| < k} w(h|\gamma|) \right], \end{aligned}$$

where

$$w(x) = \frac{e^x + \theta - 1}{\theta e^{x/\theta}} = \frac{1}{\theta} e^{x(1-1/\theta)} + \frac{\theta-1}{\theta} e^{-x/\theta}$$

is increasing in  $x \geq 0$ , and satisfies:

$$(39) \quad w(x) \geq \begin{cases} w(0) = 1, & \text{for all } x \geq 0, \\ e^{\alpha x}, & \text{for all large enough } x. \end{cases}$$

It follows from (38) that for large enough  $n$ ,

$$(40) \quad Z_n(\beta, h) \geq \mathbb{E} \left[ \theta^\ell \exp \left( \alpha h \sum_{|\gamma| \geq k} |\gamma| \right) \right],$$

which gives the claim.

For any  $\varepsilon > 0$  we have that

$$\mathbb{P}_\theta(X_n(k) > \varepsilon + \frac{1}{\alpha}(x_1^\uparrow - \frac{1}{\theta})) = \mathbb{P}_\theta \left( \alpha h \sum_{|\gamma| \geq k} |\gamma| > hn(\alpha\varepsilon + x_1^\uparrow - \frac{1}{\theta}) \right).$$

Using Markov's inequality and (37) it follows that

$$\mathbb{P}_\theta(X_n(k) > \varepsilon + \frac{1}{\alpha}(x_1^\uparrow - \frac{1}{\theta})) \leq \exp(-hn(\alpha\varepsilon + x_1^\uparrow - \frac{1}{\theta})) \frac{Z_n(\beta, h)}{Z_n(\beta, 0)}.$$

Thus

$$(41) \quad \begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_\theta(X_n(\omega) > \varepsilon + \frac{1}{\alpha}(x_1^\uparrow - \frac{1}{\theta})) \\ \leq -h(\alpha\varepsilon + x_1^\uparrow - \frac{1}{\theta}) + z(\beta, h) - z(\beta, 0) \\ = h \left( \frac{z(\beta, h) - z(\beta, 0)}{h} - \alpha\varepsilon - (x_1^\uparrow - \frac{1}{\theta}) \right). \end{aligned}$$

By Theorem 4.1 we have that  $\lim_{h \downarrow 0} \frac{z(\beta, h) - z(\beta, 0)}{h} = x_1^\uparrow - \frac{1}{\theta}$ , hence there is some  $h > 0$  such that

$$(42) \quad \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}_\theta(X_n(\omega) > \varepsilon + \frac{1}{\alpha}(x_1^\uparrow - \frac{1}{\theta})) \leq -\frac{h\alpha\varepsilon}{2}.$$

This proves the result.  $\square$

## APPENDIX A. THE TRANSPOSITION-OPERATOR

Let  $T_{xy}$  be the transposition operator (1) acting on the tensor product  $\mathcal{H}_x \otimes \mathcal{H}_y$  of two copies of  $\mathcal{H} = \mathbb{C}^{2S+1}$ . Write  $|a, b\rangle$  for the ‘uncoupled’ basis  $|a, b\rangle = |a\rangle \otimes |b\rangle$  of  $\mathcal{H}_x \otimes \mathcal{H}_y$  so that  $T_{xy}|a, b\rangle = |b, a\rangle$ . We wish to express  $T_{xy}$  in terms of the spin operators  $\mathbf{S}_x \cdot \mathbf{S}_y$ . We will show:

PROPOSITION A.1. *For each  $S \in \frac{1}{2}\mathbb{N}$  there are  $a_0, a_1, \dots, a_{2S} \in \mathbb{Q}$  such that*

$$T_{xy} = \sum_{k=0}^{2S} a_k (\mathbf{S}_x \cdot \mathbf{S}_y)^k.$$

*Proof.* We will use standard properties of additions of spins, in particular the operator

$$(43) \quad \mathbf{J} = (\mathbf{S}_x + \mathbf{S}_y)^2 = 2S(S+1)I + 2\mathbf{S}_x \cdot \mathbf{S}_y.$$

See e.g. [11, Chapter V] for background. There is an orthonormal basis for  $\mathcal{H}_x \otimes \mathcal{H}_y$  consisting of eigenvectors of  $\mathbf{J}$ , which we denote

$$|J, M\rangle \text{ for } J \in \{0, 1, \dots, 2S\} \text{ and } M \in \{-J, \dots, J\}.$$

Note that we use the notation  $|J, M\rangle$  for this basis, and  $|a, b\rangle$  for the uncoupled basis. We have

$$(44) \quad \mathbf{J}|J, M\rangle = J(J+1)|J, M\rangle.$$

The basis-change matrix from the basis  $|J, M\rangle$  to the uncoupled basis  $|a, b\rangle$  is given by the Clebsch–Gordan coefficients  $(J, M|a, b) \in \mathbb{R}$ :

$$|a, b\rangle = \sum_{J, M} |J, M\rangle (J, M|a, b).$$

These coefficients satisfy the relation:

$$(45) \quad (J, M|a, b) = (-1)^{2S-J} (J, M|b, a).$$

Write  $\mathbf{K}$  for an operator  $\mathbf{K} = \sum_{k=0}^{2S} a_k (\mathbf{S}_x \cdot \mathbf{S}_y)^k$ , where the coefficients  $a_k$  are to be chosen. Using (43) and (44) we see that

$$(\mathbf{S}_x \cdot \mathbf{S}_y)^k |J, M\rangle = x_J^k |J, M\rangle, \text{ where } x_J = \frac{1}{2}J(J+1) - S(S+1).$$

We claim that we can pick the coefficients  $a_k \in \mathbb{Q}$  so that  $\mathbf{K}|J, M\rangle = (-1)^{2S-J}$  for all  $J$ . Indeed, this holds if and only if the following matrix-equation holds:

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{2S} \\ 1 & x_1 & x_1^2 & \cdots & x_1^{2S} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{2S} & x_{2S}^2 & \cdots & x_{2S}^{2S} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{2S} \end{pmatrix} = \begin{pmatrix} (-1)^{2S} \\ (-1)^{2S-1} \\ \vdots \\ (-1)^{2S-2S} \end{pmatrix}.$$

The Vandermonde-matrix on the left is invertible and has rational entries, thus its inverse has rational entries and the claim follows.

With this choice of the  $a_k$  we have, using (45),

$$\begin{aligned}\mathbf{K}|a, b\rangle &= \sum_{J, M} \mathbf{K}|J, M\rangle(J, M|a, b\rangle) = \sum_{J, M} (-1)^{2S-J} |J, M\rangle(J, M|a, b\rangle) \\ &= |b, a\rangle,\end{aligned}$$

thus  $\mathbf{K} = T_{xy}$  as required.  $\square$

### APPENDIX B. PROOF OF LEMMA 3.1

Recall that we need to show that if  $x, y \in \Delta$  with  $\|x - y\| \leq \varepsilon < \theta^{-2}$  then  $d_H(\Delta(x), \Delta(y)) < \theta\varepsilon^{1/2}$ . Take an arbitrary  $z \in \Delta(x)$ . We will show that there is some  $z' \in \Delta(y)$  satisfying  $\|z - z'\| \leq (\theta - 1)\varepsilon^{1/2}$ . This suffices, by the symmetry between  $x$  and  $y$ .

Define  $k = \max\{j \geq 1 : z_j \geq \varepsilon^{1/2}\}$ . Since  $z_1 \geq \frac{1}{\theta} > \varepsilon^{1/2}$ , we have that  $k \geq 1$ . Also let  $\alpha = (k - 1)\varepsilon^{1/2} \in [0, 1]$ . We claim that the following choice of  $z'$  satisfies our requirements:

$$\begin{aligned}z'_1 &= \alpha + (1 - \alpha)z_1 + (1 - \alpha) \sum_{i=k+1}^{\theta} z_i, \\ z'_i &= (1 - \alpha)z_i, \quad \text{for } 2 \leq i \leq k, \\ z'_i &= 0, \quad \text{for } i > k.\end{aligned}$$

First we check that  $\|z - z'\| \leq (\theta - 1)\varepsilon^{1/2}$ . Indeed, we have that  $|z_1 - z'_1| \leq \alpha + (\theta - k)\varepsilon^{1/2} = (\theta - 1)\varepsilon^{1/2}$ , that  $|z_i - z'_i| \leq \alpha = (k - 1)\varepsilon^{1/2}$  for  $2 \leq i \leq k$ , and that  $|z_i - z'_i| = z_i < \varepsilon^{1/2}$  for  $i > k$ .

Next we check that  $z' \sqsupseteq y$ . If  $j \geq k$  then clearly  $\sum_{i=1}^j z'_i = 1 \geq \sum_{i=1}^j y_i$ . Now let  $j < k$ . Firstly, since  $z \sqsupseteq x$  and  $\|x - y\| \leq \varepsilon$ , we have that

$$\sum_{i=1}^j z_i - \sum_{i=1}^j y_i \geq \sum_{i=1}^j (x_i - y_i) \geq -j\varepsilon.$$

Hence we see that

$$\begin{aligned}\sum_{i=1}^j z'_i - \sum_{i=1}^j y_i &\geq \alpha + \left(\sum_{i=1}^j z_i - \sum_{i=1}^j y_i\right) - \alpha \sum_{i=1}^j z_i \\ &\geq -j\varepsilon + \alpha \sum_{i=j+1}^k z_i \\ &\geq -j\varepsilon + (k - 1)(k - j)\varepsilon \geq 0.\end{aligned}$$

The result follows.  $\square$

### APPENDIX C. CONVEXITY OF THE FUNCTION $R(t)$

Recall that we needed to know that a certain function  $R(t)$ , given in (33), is convex. The function  $R(t)$  is (up to a constant factor)

$$R(t) = \frac{rt \log t + (1 - rt) \log \frac{1-rt}{\theta-r} + \log \theta}{(\theta t - 1)^2}, \quad \frac{1}{\theta} < t < \frac{1}{r}.$$

Actually  $R(t)$  is well-defined for all  $t \in (0, 1/r)$ . With  $s = rt$ ,  $p = \theta/r$  and  $q = \theta/(\theta - r)$  the convexity of  $R(t)$  follows from the following result.

LEMMA C.1. For any  $p, q > 0$  satisfying  $\frac{1}{p} + \frac{1}{q} = 1$ , the function

$$f(s) = \frac{s \log(ps) + (1-s) \log(q(1-s))}{(s - \frac{1}{p})^2}, \quad s \in (0, 1)$$

is convex.

*Proof.* Let  $g(s) = (s - \frac{1}{p})^4 f''(s)$ . Direct computation gives

$$g(s) = \frac{(s - \frac{1}{p})^2}{s(1-s)} + (2s + \frac{4}{p}) \log(ps) + (6 - (2s + \frac{4}{p})) \log(q(1-s))$$

and thus

$$g''(s) = \frac{2(s - \frac{1}{p})^2(1 - 3s + 3s^2)}{s^3(1-s)^3} \geq 0.$$

Hence  $g$  is convex on  $(0, 1)$ . It is easy to see that  $g(s) \rightarrow +\infty$  as  $s \downarrow 0$  or  $s \uparrow 1$ , so  $g$  has a unique minimum in  $(0, 1)$ . We have that

$$\begin{aligned} g'(s) &= 2 \log(ps) - 2 \log(q(1-s)) \\ &\quad + \frac{\frac{4}{p} + 2s}{s} + \frac{(s - \frac{1}{p})^2}{s(1-s)^2} - \frac{(s - \frac{1}{p})^2}{s^2(1-s)} \\ &\quad + 2 \frac{s - \frac{1}{p}}{s(1-s)} - \frac{6 - \frac{4}{p} - 2s}{1-s}, \end{aligned}$$

and therefore  $g'(\frac{1}{p}) = 0$ . It follows that

$$g(s) \geq g(\frac{1}{p}) = 0 \text{ for all } s \in (0, 1).$$

We conclude that  $f''(s) \geq 0$  for all  $s \in (0, 1)$ , as required.  $\square$

**Acknowledgement.** Part of this work was carried out while the author was at Chalmers University of Technology in Göteborg, Sweden, with support from the Knut and Alice Wallenberg Foundation.

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